

On the use of the Operator Product Expansion to constrain the hadron spectrum.

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Abstract

We call into question a recently proposed idea on how the short-distance behavior of QCD correlation functions constrains hadronic states to become degenerate parity eigenstates as one goes up in the spectrum. In particular, we point out that the sum rules which have been proposed in this context are in general regulator dependent, and thus ill-defined.

In QCD with an infinite number of colors, meson resonances are infinitely narrow and, consequently, can be precisely identified as quantum states. Furthermore, because of asymptotic freedom and confinement, the number of meson resonances in each channel is infinite [1]. Given an operator with the appropriate quantum numbers (such as a (partially) conserved current), each state is characterized by two parameters, its mass, and the amplitude of the operator between this state and the vacuum. In the absence of a solution to large- N_c QCD, it is interesting to see how much the values of these parameters can be constrained on the basis of large- N_c , chiral symmetry, and the Operator Product Expansion (OPE).

An interesting attempt to do this was presented in Ref. [2]. The main result of this analysis is that vector and axial-vector mesons would have to pair up as one goes up in the spectrum so that chiral symmetry is restored at higher energies. In particular, two sum rules similar to those derived by Weinberg long ago [3] were proposed, and applied to various models of the meson spectrum in the (non-singlet) vector and axial-vector channels. These sum rules were derived in two complementary ways. The first derivation is based only on very general properties of the OPE, and the second one uses only chiral symmetry and null-plane charges.¹

These results of Ref. [2] have been used to infer the spectrum of hybrids [4], to “test” models (such as that of Ref. [5]), and to argue for the phenomenon of chiral symmetry restoration for highly excited states in the hadronic spectrum [7, 8, 9]. Since these claims are rather far-reaching, we consider it indispensable to have a closer look at the theoretical foundations on which the analysis of Ref. [2] is based.

Let us start by defining the (covariantly time-ordered) vector and axial-vector two-point functions in the chiral limit as

$$\Pi_{V,A}^{\mu\nu} = i \int d^4x e^{iqx} \langle 0 | T(J_{V,A}^\mu(x) J_{V,A}^{\nu\dagger}(0)) | 0 \rangle = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi_{V,A}(q^2) , \quad (1)$$

with $J_V^\mu(x) = \bar{d}(x) \gamma^\mu u(x)$ and $J_A^\mu(x) = \bar{d}(x) \gamma^\mu \gamma^5 u(x)$. Both functions $\Pi_{V,A}(Q^2 \equiv -q^2)$ satisfy the dispersion relation (up to one subtraction)

$$\Pi_{V,A}(Q^2) = \int_0^\infty \frac{dt}{t + Q^2} \frac{1}{\pi} \text{Im} \Pi_{V,A}(t) . \quad (2)$$

In large- N_c QCD one finds that

$$\frac{1}{\pi} \text{Im} \Pi_V(t) = 2 \sum_{n=1}^{N_V} F_V^2(n) \delta(t - M_V^2(n)) , \quad (3)$$

$$\frac{1}{\pi} \text{Im} \Pi_A(t) = 2 \sum_{n=1}^{N_A} F_A^2(n) \delta(t - M_A^2(n)) , \quad (4)$$

where $F_{V,A}(n)$ and $M_{V,A}(n)$ are the two parameters for each resonance n in each channel (V or A) mentioned above, with $n = 1$ in the axial channel corresponding to the pion.²

¹See also Ref. [6]

²Therefore, $F_A^2(n=1) \equiv F_\pi^2 = (93 \text{ MeV})^2$ and $M_A^2(n=1) \equiv M_\pi^2 = 0$ in the chiral limit, to which we will restrict ourselves in this paper.

It follows that

$$\Pi_V(Q^2) = \sum_{n=1}^{N_V} \frac{F_V^2(n)}{Q^2 + M_V^2(n)} , \quad (5)$$

$$\Pi_A(Q^2) = \sum_{n=1}^{N_A} \frac{F_A^2(n)}{Q^2 + M_A^2(n)} , \quad (6)$$

where it is understood that $N_{V,A}$ are to be taken to infinity after physical observables have been calculated, with $Q^2 \ll M_V^2(N_V), M_A^2(N_A)$. We thus emphasize the need for an ultraviolet cutoff in these sums.

At large values of Q^2 , $\Pi_{V,A}$ in Eqs. (5,6) have to reproduce the parton-model logarithm. This follows easily with the Euler-Maclaurin summation formula, which, applied to our case, reads

$$\begin{aligned} \Pi_{V,A}(Q^2) = & \int_0^{N_{V,A}+1} dn \frac{F_{V,A}^2(n)}{Q^2 + M_{V,A}^2(n)} \\ & - \frac{1}{2} \left\{ \frac{F_{V,A}^2(0)}{Q^2 + M_{V,A}^2(0)} + \frac{F_{V,A}^2(N_{V,A}+1)}{Q^2 + M_{V,A}^2(N_{V,A}+1)} \right\} + \dots \quad . \end{aligned} \quad (7)$$

This indeed reproduces a “log Q^2 ” when $Q^2 \rightarrow \infty$ (as long as we take $N_{V,A}$ to infinity first) if one assumes that³

$$M_{V,A}^2(n) \sim n , \quad F_{V,A}^2(n) \sim F_{V,A}^2 \quad , \quad (8)$$

for $n \rightarrow \infty$, with $F_{V,A}^2$ constants independent of n . This assumption is for example in accord with Regge Theory in which the “daughter trajectories” are given by

$$M_{V,A}^2(n) = \Lambda_{V,A}^2 n + \text{constant} . \quad (9)$$

As a matter of fact, the behavior of Eq. (8) is what is obtained in two dimensions, for which large- N_c QCD can be solved [12]. It is reasonable to expect that Eq. (8) maybe true also in four dimensions.

We know from perturbation theory that the coefficient of the logarithm in Π_V and Π_A is the same. This leads to the further condition

$$\frac{F_V^2}{\Lambda_V^2} = \frac{F_A^2}{\Lambda_A^2} = \frac{N_c}{24\pi^2} . \quad (10)$$

Furthermore, we know that $\Pi_{LR}(Q^2) = \Pi_V(Q^2) - \Pi_A(Q^2)$ should vanish for $Q^2 \rightarrow \infty$, in order to avoid conflict with chiral symmetry and asymptotic freedom. It follows directly from Eq. (7) that this is only the case if the limits $N_{V,A} \rightarrow \infty$ are taken such that

$$\frac{N_A}{N_V} \rightarrow \frac{\Lambda_V^2}{\Lambda_A^2} . \quad (11)$$

³Note that in general $F_{V,A}^2(n) \sim \frac{d}{dn} M_{V,A}^2(n)$ would do just as well, so Eq. (8) is not the only possibility.

One should keep in mind that there is a certain amount of arbitrariness in the exact relation between N_V and N_A . Clearly, physical results have to be *independent* of this arbitrariness in order to guarantee universality of physical quantities. In other words, provided Eq. (11) is satisfied, no additional information is needed for how the limit $N_{V,A} \rightarrow \infty$ is taken in Eqs. (5,6). This point will be crucial in what follows.

Equations (5) and (6) each contain sums over an infinite number of states, and the expansion for large values of Q^2 therefore is non-trivial. In particular, it does not commute with the operation of summing over n . However, Ref. [2] appears to argue that this is only true of the individual sums in Π_V and Π_A , because of the presence of the parton model $\log Q^2$. Consequently, in the difference $\Pi_V - \Pi_A$, which does not contain this logarithm, the claim is that one recovers the OPE as a naive expansion in inverse powers of Q^2 , which commutes with the operation of summing over n . According to Ref. [2], one then finds at large Q^2 (with $N_{V,A} \rightarrow \infty$)

$$\begin{aligned} \Pi_V - \Pi_A &\sim \frac{\left\{ \sum_{n=1}^{N_V} F_V^2(n) - \sum_{n=1}^{N_A} F_A^2(n) \right\}}{Q^2} \\ &+ \frac{\left\{ \sum_{n=1}^{N_V} F_V^2(n) M_V^2(n) - \sum_{n=1}^{N_A} F_A^2(n) M_A^2(n) \right\}}{Q^4} \\ &+ \mathcal{O}\left(\frac{1}{Q^6}\right) . \end{aligned} \quad (12)$$

In particular, then (again with $N_{V,A} \rightarrow \infty$),

$$\sum_{n=1}^{N_V} F_V^2(n) - \sum_{n=1}^{N_A} F_A^2(n) = 0 , \quad \text{and} \quad (13)$$

$$\sum_{n=1}^{N_V} F_V^2(n) M_V^2(n) - \sum_{n=1}^{N_A} F_A^2(n) M_A^2(n) = 0 , \quad (14)$$

from the absence of operators of dimension 2 and dimension 4 in the OPE of $\Pi_V - \Pi_A$. The claim of Ref. [2] seems to be that these equations are well-defined, regulator-independent large- N_c QCD sum rules and, thus, that these sum rules can be used to restrict *ansätze* for the hadronic spectra.

These claims are not justified. Eqs. (13,14) must be supplied with information on how $N_{V,A}$ go to infinity, because, as we shall see, the results actually depend on this. Since the initial Eqs. (5, 6) are insensitive to how this limit is taken (as long as Eq. (11) is satisfied), this information can only be supplied by the hadronic spectrum itself, which consequently makes Eqs. (13,14) not very useful for constraining that spectrum.

Of course all this happens because, in order for the operations of summing over n and expanding in $1/Q^2$ to commute, all sums involved would need to be absolutely and uniformly convergent, and this is clearly not the case. In particular, the absence of the parton model logarithm cannot be a convincing reason for an expansion such as Eq. (12) to make sense. The Adler function, which is defined as

$$-Q^2 \frac{d}{dQ^2} \Pi_V(Q^2) , \quad (15)$$

by construction does not have a logarithm either, but it is easy to convince oneself that it does not need to have an OPE with numerators given by sums over resonance parameters [16, 15].

As a matter of fact, there are at least three objections to the claims of Ref. [2]. First, the claim about chiral symmetry restoration for highly excited mesons does *not* take place in the only situation where large- N_c QCD is soluble and everything is under good theoretical control, which is the case of two dimensions [12, 13].⁴ There, one finds that the spectrum of meson masses, which for highly excited states goes like

$$M_n^2 \sim \text{constant} \times n, \quad (16)$$

actually *alternates* in parity as one increases n by one unit. The conflict with the claims of Ref. [2] can be made most obvious if one considers scalar and pseudo-scalar two-point functions $\Pi_{S,P}$. This is due to the fact that $\Pi_S(Q^2)$ and $\Pi_P(Q^2)$ in two dimensions have representations like those of $\Pi_{V,A}(Q^2)$ in the ordinary four-dimensional case,⁵ i.e. Eqs. (2,3,4,8). A rerun of the analysis of [2] in two dimensions for $\Pi_{S,P}$ would conclude that the spectrum of excited scalar and pseudo-scalar mesons should be degenerate; however, clearly, it is not. Two-dimensional large- N_c QCD is therefore a counter example to the claim of Ref. [2].

Second, the functional dependence of Eq. (12) on Q^2 is not correct. The currents in the correlator Eq. (1) have no anomalous dimensions. The leading operator in its OPE expansion is the quark condensate (squared) which has non-zero anomalous dimensions even in the large- N_c limit. The corresponding Wilson coefficient therefore has a residual $\log Q^2$ dependence which is missing in Eq. (12) [10]. Therefore Eq. (12) cannot be exact. This was the point of view taken by the authors of Refs. [11] in constructing an *approximation* to large- N_c QCD with a *finite* number of resonances. Clearly, in this case all expansions at large Q^2 make definite sense [16].

If one considers Eq. (12) as some approximate statement, up to such offending logs, then our third objection becomes relevant. This has to do with the fact that these sums are ill-defined because, unlike those in Ref. [11], they are infinite and, as we already pointed out, must be regulated. This is why we wrote Eqs. (13,14) with sums cut off at N_V and N_A . It is then important to remember that these sums should not depend on the precise value taken for the cutoffs N_V and N_A and, for instance, the same physical results should be obtained choosing $N_V + a$ instead of N_V for arbitrary finite a in the limit $N_V \rightarrow \infty$. This is a fundamental property which guarantees the universality of the physics in any Quantum Field Theory and, in fact, one can readily check that the original expressions of Eq. (7) satisfy this requirement. However, the sum rules in Eqs. (13,14) *do* depend on a . Let us take Eq. (13), for example, and replace $N_V \rightarrow N_V + a$ in Eq. (13). Using again the Euler-Maclaurin summation formula one

⁴Note that two-dimensional large- N_c QCD in the chiral limit does exhibit spontaneous breaking of chiral symmetry, if the chiral limit $m_{quark} \rightarrow 0$ is taken after the limit $N_c \rightarrow \infty$. This is precisely the order of limits in which 't Hooft solved this theory and does not contradict Coleman's theorem. For details, see Ref. [13] and Refs. therein.

⁵There are several reasons why $\Pi_{S,P}$ are actually the natural two-dimensional analogs of $\Pi_{V,A}$ in four dimensions [14].

obtains:

$$\begin{aligned} \sum_{n=1}^{N_V+a} F_V^2(n) - \sum_{n=1}^{N_A} F_A^2(n) &\approx \int_0^{N_V+a-1} dn F_V^2(n) - \int_0^{N_A-1} dn F_A^2(n) + \dots \\ &\approx F_V^2(N_V + a - 1) - F_A^2(N_A - 1) + \dots \end{aligned} \quad (17)$$

where $F_{V,A}$ are given in Eq. (8) and the ellipsis are terms subleading in $N_{V,A}$ as $N_{V,A} \rightarrow \infty$. Use of Eqs. (10,11) then shows the cancellation of terms leading in the cutoff Λ in the above expression, but one also sees that the sum rule is a dependent. This simply means that this sum rule is not consistent with universality.

Let us study the issue of regulator dependence in some more detail, using a recently analyzed *ansatz* for the vector and axial-vector meson spectrum [5] as an example. In fact, a claim of Ref. [2] is that the sum rules of Eqs. (13,14) rule out this model. To illustrate the point about regulator dependence made above, let us show how one can choose to satisfy Eq. (13), by tuning the cutoff dependence in our model [5], without any chiral symmetry restoration at all.

In the model of Ref. [5], all $F_A(n)$ were taken equal, $F_A(n) = F_A$, except for the pion, with $F_A(1) = F_\pi$, as were all $F_V(n) = F_V$, except that of the lowest vector resonance, the ρ , for which $F_V(1) = F_\rho$. Masses were taken to follow the Regge-like pattern

$$\begin{aligned} M_V^2(1) &= M_\rho^2, \\ M_V^2(n) &= m_V^2 + (n-2)\Lambda_V^2, \quad n > 1, \\ M_A^2(1) &= M_\pi^2 = 0, \\ M_A^2(n) &= m_A^2 + (n-2)\Lambda_A^2, \quad n > 1, \end{aligned} \quad (18)$$

where $m_{V,A}$ and $\Lambda_{V,A}$ are four parameters with dimension of mass. In Ref. [5] various relations generalizing the Weinberg sum rules were derived from the OPE between the parameters of the model, $m_{V,A}$, $\Lambda_{V,A}$, $F_{V,A}$, M_ρ , F_ρ and F_π . As we already emphasized, there is a certain freedom in the choice of the exact relation between the cutoffs N_V and N_A , which can be made explicit by choosing⁶

$$\Lambda^2 \equiv N_A \Lambda_A^2 = N_V \Lambda_V^2 + \mu^2, \quad (19)$$

where μ is a finite, but otherwise undetermined parameter. The only requirement is that physically meaningful results should not depend on μ , and this is precisely where the sum rules in Eqs. (13,14) run into trouble.⁷ Let us show this by applying them to the *ansatz* of Ref. [5]. From Eq. (13), we obtain for this *ansatz*

$$F_\rho^2 + (N_V - 1)F_V^2 - (N_A - 1)F_A^2 = F_\pi^2. \quad (20)$$

⁶In Ref. [5] a specific choice was made for these parameters, but it is straightforward to show that none of the results obtained there depend on their values.

⁷It is important to recall that in the application of the OPE the scale Λ has to be taken much larger than any other scale in the problem, including the (euclidean) momentum Q flowing through the current-current correlators.

Using Eqs. (19) and (10) this can be rewritten as

$$\frac{N_c}{24\pi^2} \mu^2 = -F_\pi^2 + F_A^2 + F_\rho^2 - F_V^2. \quad (21)$$

If μ satisfies this condition, Eq. (13) is satisfied. This demonstrates the problem with the sum rule Eq. (13): it depends on the choice of the values of the unphysical parameter μ whether the sum rule will be obeyed or not. The divergences in both sums (*i.e.* the terms linear in N_V and N_A in Eq. (20)) do cancel as they should, but there are “left-over” finite terms which take on arbitrary values depending on the details of the regularization.

Something similar also happens with the sum rule Eq. (14). Substituting our *ansatz* leads to *two* constraints on the parameter μ , one from the requirement that the linear term in Λ^2 vanish, and one from the finite part. (The leading terms, quadratic in Λ^2 , cancel identically.) These two constraints turn out to be incompatible with Eq. (21), unless one requires the full vector and axial-vector spectral functions to be equal, in which case one finds that choosing $\mu = 0$ satisfies all constraints (not surprisingly).

This latter conclusion is precisely that advocated in Ref. [2]. However, our discussion makes it clear that this is misleading: whether each individual sum rule is satisfied or not depends on the detailed choice of the finite parts of the regulator, represented here by the unphysical parameter μ . Consequently, the choice made in Ref. [2] is arbitrary and ad hoc, not based on chiral symmetry, and not on any other known property of QCD. What it does, in fact, is arbitrarily enforcing the degeneracy of the spectrum, rather than obtaining it as a result. In contrast, all sum rules derived in Ref. [5] are independent of the value of μ . The sum rules of Eqs. (13) and (14) will only be independent of the regulator if each of the sums in these equations converges, and their derivation only holds if this is the case. The real spectrum of QCD is unlikely to satisfy this requirement, in which case the sum rules cease to be meaningful.

We have already indicated why we believe that the derivation of these sum rules from the OPE is flawed. In addition to the OPE-based argument, another derivation was given in Ref. [6]. There, chiral symmetry was implemented on finite representations of the $SU(2)_L \times SU(2)_R$ algebra in the null plane. However, it is clear that what is needed is the implementation of chiral symmetry on infinite representations, and one expects that consequently sums over states will again have to be properly regularized. This was not done in Ref. [6], making this derivation equally dubious.

It would of course be very interesting to infer knowledge about the hadron spectrum from the OPE, as one does expect that the higher-lying spectrum should be constrained by the OPE. Unfortunately, in our opinion, all one can confidently say is that the large- k behavior of the coefficients c_k in the OPE

$$\Pi_V(Q^2) - \Pi_A(Q^2) \sim \sum_k \frac{c_k}{Q^{2k}}, \quad (22)$$

is correlated with the spectrum of highly excited states [13, 15, 16]. Following Zhitnitsky [13], we write

$$\Pi_V(Q^2) - \Pi_A(Q^2) = \sum_{n=1}^{\infty} \left(\frac{F_V^2(n)}{Q^2 + M_V^2(n)} - \frac{F_A^2(n)}{Q^2 + M_A^2(n)} \right)$$

$$= \int_0^\infty \frac{dt}{e^t - 1} f(t, Q^2) \quad , \quad (23)$$

where the function $f(t, Q^2)$ satisfies

$$\frac{F_V^2(n)}{Q^2 + M_V^2(n)} - \frac{F_A^2(n)}{Q^2 + M_A^2(n)} = \int_0^\infty dt e^{-nt} f(t, Q^2) . \quad (24)$$

This equation can be inverted with the help of the inverse Laplace transform to read

$$f(t, Q^2) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{-iT}^{+iT} dn e^{nt} \left(\frac{F_V^2(n)}{Q^2 + M_V^2(n)} - \frac{F_A^2(n)}{Q^2 + M_A^2(n)} \right) . \quad (25)$$

For the function f to be uniquely determined, one needs to specify the interpolation of the left-hand side of Eq. (24) to non-integer values of n . Here we have in mind that, at least for large n , $M_{V,A}^2(n) = \Lambda_{V,A}^2 n$ and $F_{V,A}^2(n) = F_{V,A}^2$ for all positive n . The singularities of the integrand lie to the left of the integration path, as they should. Using that

$$\frac{t}{e^t - 1} = \sum_{k=0}^\infty \frac{B_k}{k!} t^k \quad , \quad \text{with} \quad B_k \sim \frac{(2k)!}{2^{2k-1} \pi^{2k}} \left(1 + \frac{1}{2^{2k}} + \dots \right) , \quad (26)$$

where the B_n are Bernoulli's numbers, one obtains the asymptotic expansion

$$\Pi_V(Q^2) - \Pi_A(Q^2) \sim \sum_{k=0}^\infty \frac{B_k}{k!} \int_0^\infty dt t^{k-1} f(t, Q^2) . \quad (27)$$

In order to calculate the integral in Eq. (25) one completes the path with a semi-circular contour to the left of the imaginary axis, and of radius T . The behavior of $f(t, Q^2)$ for large values of Q^2 is dominated by the asymptotic behavior of $F_{V,A}(n)$ and $M_{V,A}^2(n)$ for large n , and one finds

$$f(t, Q^2) \sim \left(\frac{F_V^2}{\Lambda_V^2} e^{-\frac{Q^2}{\Lambda_V^2} t} - \frac{F_A^2}{\Lambda_A^2} e^{-\frac{Q^2}{\Lambda_A^2} t} \right) \quad (28)$$

as Q^2 becomes large.⁸ Inserting this into Eq. (27) one finally obtains that, for large k , the coefficients of the OPE in Eq. (22) behave like

$$c_k \sim \frac{(2k)!}{2^{2k-1} k \pi^{2k}} \left(F_V^2 \Lambda_V^{2(k-1)} - F_A^2 \Lambda_A^{2(k-1)} \right) . \quad (29)$$

Equation (29) expresses the difficulty. One would need to know the asymptotic behavior of the coefficients c_k in large- N_c QCD to be able to infer whether $\Lambda_V = \Lambda_A$ and $F_V = F_A$ or not, *i.e.* whether chiral symmetry is restored for highly excited hadronic states.

In summary, the claims made in Refs. [2, 7, 8] to the effect that the OPE constrains the spectrum of highly excited hadrons to be degenerate parity eigenstates are

⁸Note that the integral in Eq. (27) converges for $k = 0$ because of Eq. (10).

unjustified. There may be some indication [9] of chiral restoration in the current experimental data on the spectrum of scalars and pseudoscalars [17], but drawing any definite conclusions at this stage seems premature. In our opinion, the question of chiral symmetry restoration for very highly excited hadrons is a very interesting one which, however, remains completely open and deserves further investigation.

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